HYDRODYNAMICS OF A QUASI-ONE-DIMENSIONAL PACKET

G. A. Kuz'min and A. Z. Patashinskii

UDC 532.517.4

To describe turbulence as coherent-structure chaos [1], one needs to establish the structure types. Some information has accumulated on structures whose scales ℓ are of the order of or greater than the basic turbulence scale L. There are models in which the moving elements are vortex rings and filaments or shear layers [1, 2].

Very scanty experimental evidence is available on the characteristic structural elements on small scales $\ell << L$. We merely know that small-scale motions are highly variable and localized in regions with small relative volumes. One assumes that the small-scale motions may have some organization. There is a combination of marked nonlinearity in the motion within a small volume and the extracting and orienting action of the large-scale velocity, which can lead to considerable ordering in the motion in that volume. Such ordering in particular is likely for motions from the energy-dissipation range, in which there is relaxation in the motions generated by the inertial range.

Here we show that the turbulence parameters in the energy-dissipation range are determined by quasi-one-dimensional packets of hydrodynamic harmonics. Equations are derived for the packet evolution over time and the characteristic features have been established for the solutions that are important to understanding the nonlinear dynamics of the pulsations in the dissipative scale range.

Turbulence Parameters in the Energy-Dissipation Range. The local structure in developed turbulence is determined by length scales: the scale of the treatment L and the Kolmogorov dissipation scale η. In the inertial scale range $\eta \ll l \ll L$, there is nonlinear energy transfer from the large scales to the small ones, while the viscosity is not important. Conversely, for $\ell < \eta$, the viscosity is important, since energy dissipation occurs in that range. A special study is needed to establish the role of nonlinearity for $\ell < \eta$.

The assumption that the dissipating harmonics have weak nonlinearity implies exponential decay in the turbulence spectrum for wave numbers $k >> \eta^{-1}$ [2]:

$$E(k) \sim \exp\,[-(\eta k)^2]. \tag{1}$$

A more detailed theoretical analysis shows that there may be reasons why E(k) decreases more slowly than in (1). The first is that there are functions in η, which are due to fluctuations in the energy influx from the inertial range. Averaging E over the η fluctuations can lead to E(k) decreasing more slowly than in (1) [3, 4]. A discussion of the small-scale structure statistics lies outside the scope of the present paper. Another factor is the marked nonlinearity in the dissipating harmonics, which then incorporated in field-theory methods gives the turbulence spectrum for $\eta k >> 1$ [5-7]:

$$E(k) \sim \exp\,(-\eta k). \tag{2}$$

The (2) asymptote is provided by simple dynamic models: the Langevin nonlinear equation, Burgers equation, and the Lorentz model [4]. Here η is related to the distance of the singularity in the fluctuating function closest to the real axis. If the fluctuations are bounded, the (2) asymptote persists on averaging over the fluctuations. The application of that method to hydrodynamic turbulence is limited by the multiple dimensions.

We show that the pulsation dynamics for $\eta k >> 1$ will be quasi-one-dimensional. The Navier–Stokes equations for an incompressible liquid in the Fourier representation in terms of the spatial coordinates are

---

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 98-101, March-April, 1990. Original article submitted December 26, 1988.

$$(\partial/\partial t + \nu k^2) u_i(\mathbf{k}, t) = -i/2 P_{ijl}(\mathbf{k}) \int d^3q u_j(\mathbf{q}, t) u_l(\mathbf{k}-\mathbf{q}, t); \tag{3}$$

$$\mathbf{k}\mathbf{u}(\mathbf{k}, t) = 0$$

$$(P_{ijl}(\mathbf{k}) = k_j \Delta_{il}(\mathbf{k}) + k_l \Delta_{ij}(\mathbf{k}),\ \Delta_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2). \tag{4}$$

One naturally assumes that the characteristic harmonic amplitude for $\eta k \gg 1$ is exponentially dependent on the wave number

$$\mathbf{u}(\mathbf{k}) \sim \exp[-(\eta k)^\gamma] \tag{5}$$

($\gamma > 0$). We substituted (5) on the right in (3), which shows that for $\gamma < 1$, the main contribution comes from the region where $q \ll |\mathbf{k}-\mathbf{q}| \sim k$ or $|\mathbf{k}-\mathbf{q}| \ll q \sim k$, so there is direct energy transfer from the harmonics with scale η to the $\mathbf{u}(\mathbf{k})$ ones. The (3) equations are linearized and give the (1) spectrum [2], which does not agree with the assumption $\gamma < 1$. For $\gamma > 1$, the most important contribution is from the $q \sim |\mathbf{k}-\mathbf{q}| \sim k/2$ range. The right-hand side in (3) is of the order of $P_{ijl}(\mathbf{k}) u^2(\mathbf{k}/2) k^3 \sim \exp[-(\eta k)^\gamma 2^{1-\gamma}]$. In the limit $\eta k \to \infty$, the right-hand and left-hand parts of the equation coincide as to order of magnitude only for $\gamma = 1$. Then the largest contribution to the integral comes from the region in which the wave vectors $\mathbf{q}$ and $\mathbf{k}$ are almost collinear. The contribution from exactly collinear $\mathbf{q}$ and $\mathbf{k}$ becomes zero because of the $P_{ij\ell}(\mathbf{k})$ factor, while that from noncollinear $\mathbf{q}$ and $\mathbf{k}$ is exponentially small with respect to the transverse deviations. The phases of the $\mathbf{u}(\mathbf{k})$ should be correlated. The contribution from the incoherent component on the right in (3) is small because of the fast oscillations.

We conclude that the turbulence parameters in the extreme shortwave region $\eta k \gg 1$ are determined by coherent nonlinear harmonic packets having almost collinear wave vectors. We now derive approximate one-dimensional equations for such a packet, which follow from (3) and (4) after expansion in terms of the small noncollinearity.

Dynamic Equations for a Harmonic Packet. The complete information on a packet having almost collinear wave vectors $\mathbf{k}$ is contained in the set of linear moments for $\mathbf{u}(\mathbf{k})$:

$$\theta^i_{i_1 \ldots i_n}(p) = \int_\sigma \varkappa_{i_1} \varkappa_{i_2} \ldots \varkappa_{i_n} u_i(p\mathbf{e} + \boldsymbol{\varkappa})\, d^2\varkappa, \tag{6}$$

in which n = 0, 1, 2, ...; and e is unit vector along the packet axis, while $\boldsymbol{\varkappa}$ is the component of the wave vector perpendicular to that axis, and p is the longitudinal projection of the wave vector. The integration is in the σ plane perpendicular to the axis.

Here we consider the zeroth and first moments, which provide the main information on the packet structure. The higher-order moments describe the fine structure and govern the contributions to the effective viscosity and interaction unimportant on the dissipative scale.

We get the dynamic equations for θ^i and θ^i_μ from (3) and (4) after integrating them in the σ plane with weighting factors 1 and $\varkappa_\mu$. One can expand $P_{ij\ell}$ in the integral as a series in the deviations $\varkappa = \mathbf{k} - p\mathbf{e}$:

$$\begin{aligned} P_{ijl}(\mathbf{k}) &= P_{ijl}(p\mathbf{e}) + \partial P_{ijl}(\mathbf{k})/\partial k_\alpha |_{\varkappa=0} \varkappa_\alpha + \ldots \\ &= p P_{ijl}(\mathbf{e}) + \varkappa_j[\Delta_{il}(\mathbf{e}) - e_i e_l] + \varkappa_l[\Delta_{ij}(\mathbf{e}) - e_i e_j] - 2\varkappa_i e_j e_l + \ldots \end{aligned} \tag{7}$$

Then (7) enables one to replace (3) and (4) by a system for the linear moments $\theta^i_{i_1 \ldots i_n}$.

The (4) incompressibility equations give a set of kinematic relations:

$$p e_j \theta^j_{i_1 \ldots i_n} + \theta^j_{j, i_1, \ldots, i_n} = 0 \quad (n = 0, 1, 2, \ldots). \tag{8}$$

If we neglect the moments above the first order, then

$$p e_j \theta^j(p) + \theta^j_j(p) = 0,\ e_j \theta^j_i = 0. \tag{9}$$

Analogous integration of (3) in the σ plane gives dynamic equations for θ^i, θ^i_m:

$$\partial\theta^i(p)/\partial t + \nu p^2\theta^i(p) = -ip/2P_{ijl}(\mathrm{e})\int dq\theta^j(q)\theta^l(p-q)$$
$$-i\int dq\,\{[\theta^j_j(q)\theta^l(p-q) + \theta^j(q)\theta^l_j(p-q)][\Delta_{il}(\mathrm{e}) - e_ie_l]$$
$$-2\theta^j_i(q)\theta^l(p-q)e_je_l\} + \ldots; \tag{10}$$
$$\partial\theta^i_m(p)/\partial t + \nu p^2\theta^i_m(p) = -ip/2\int dq\,[\theta^i_m(q)\theta^l(p-q)$$
$$+\theta^j(q)\theta^l_m(p-q)][e_j\Delta_{il}(\mathrm{e}) + e_l\Delta_{ij}(\mathrm{e})]. \tag{11}$$

One can reduce (10) and (11) to a system of differential equations in one-dimensional space if one performs an inverse Fourier transformation with respect to the longitudinal wave number. If we take the x axis as the packet axis and introduce

$$f(x) = e_j\int \exp(ipx)\theta\ (p)\,dp = u_x(x, 0, 0); \tag{12}$$

$$h_i(x) = \Delta_{im}(\mathrm{e})\int \exp(ipx)\theta^m(p)\,dp = \Delta_{im}(\mathrm{e})\,u_m(x, 0, 0); \tag{13}$$

$$g_{im}(x) = -i\Delta_{il}(\mathrm{e})\int \exp(ipx)\theta^l_m(p)\,dp = \Delta_{il}(\mathrm{e})\,\Delta_{mn}(\mathrm{e})\frac{\partial u_l}{\partial x_n}(x, 0, 0), \tag{14}$$

(9)-(11) become

$$g_{jj} = \partial f/\partial x; \tag{15}$$

$$\partial f/\partial t + f\partial f/\partial x = \nu\partial^2 f/\partial x^2; \tag{16}$$

$$\partial g_{im}/\partial t + \partial(fg_{im})/\partial x = \nu\partial^2 g_{im}/\partial x^2; \tag{17}$$

$$\partial h_i/\partial t + \partial(fh_i)/\partial x = \nu\partial^2 h_i/\partial x^2 + g_{ij}h_j + g_{jj}h_i. \tag{18}$$

Moment Solution Features. The complete solution to (16)-(18) can be found by solving (16), (17), and (18) sequentially. We first determine the (12) longitudinal velocity, which satisfies the Burgers equation (16). For a known f, (17) is linear in the g_{im} tensor. If the solution to that equation has been derived, it remains to solve the linear equation (18).

The only nonlinear equation in the system, the Burgers equation, can be integrated analytically [8]. The solutions are characterized by a tendency to give rise to shock fronts. The positions and intensities of those fronts are determined by features in the analytic function f(x) in the plane of the complex variable $x = \zeta_1 + i\zeta_2$ [4]. The singularity closest to the real axis defines the asymptote for the Fourier harmonics in the large wave number range.

The g_{im} tensor satisfies (17); the right-hand side describes the diffusion effect from the viscosity, while the $\partial(fg_{im})/\partial x$ term describes the convective transport of g_{im} by the field f. The g_{im} solutions have a characteristic tendency for the moduli of the components to increase at points where $\partial f/\partial x$ is negative. An opposite tendency is for the narrow peaks to spread by diffusion because of the viscosity effect.

From (15), the trace of the tensor g_{ij} is found by differentiating f with respect to x, so it remains to derive the trace-free component $g'_{im} = g_{im} - \delta_{im}g_{jj}/3$. The symmetrical component $(g'_{im} + g'_{mi})/2$ is the strain tensor, while the antisymmetric one is expressed in terms of the longitudinal vorticity $(g_{im} - g_{mi})/2 = \Delta_{il}(\mathrm{e})\Delta_{mn}(\mathrm{e})e_{lnj}\omega_j$. The (13) transverse velocity satisfies (18). The last two terms on the right in (18) give the convective influx of transverse momentum at the packet axis.

We see from (16)-(18) that there are particular solutions in which any components of f, g_{im}, and h_i are identically zero. As the nonlinearity is important only for $f \neq 0$, the main interest attaches to the behavior of the longitudinal velocity f. That conclusion conflicts with the common assumption that the spectrum asymptote for $\eta k \to \infty$ is determined by the stretching effect for the vortex lines, namely by the $\partial(fg_{im})/\partial x$ term on the left in (17). If accidently the region with considerable vorticity falls within the front for function f, the longitudinal vorticity is amplified. However, the above implies that this is a secondary effect from the viewpoint of calculating the spèctrum asymptote for $\eta k \to \infty$.

These equations apply if the Reynolds number Re for a packet is small. Formal application to packets with large Re leads to an inertial interval in the Burgers equation with spectrum k^{-2}, which differs from the Kolmogorov–Obukhov spectrum. One cannot use (16)-(18) for a packet with large Re because such a packet is unstable with respect to the generation of harmonics having large transverse wave-vector components, so it is rapidly destroyed.

LITERATURE CITED

1. A. K. Hussain, "Coherent structures – reality and myth," Phys. Fluids, 26, No. 10 (1983).
2. A. S. Monin and A. M. Yaglom, Statistical Hydromechanics [in Russian], Nauka, Moscow (1967).
3. B. S. Keller and A. M. Yaglom, "The effects of energy-dissipation fluctuations on the spectrum shape in the extreme short-wave region," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 3 (1970).
4. U. Frisch and R. Morf, "Intermittency in nonlinear dynamics and singularities at complex times," Phys. Rev. A. Gen. Phys., 5, No. 6 (1981).
5. R. H. Kraichnan, "The structure of isotropic turbulence at very high Reynolds number," J. Fluid Mech., 5, No. 6 (1959).
6. G. A. Kuz'min and A. Z. Patashinskii, "The short-wave asymptote for a turbulence spectrum," Zh. Éksp. Teor. Fiz., 76, No. 6 (1979).
7. M. M. Dubovikov and V. I. Tatarskii, "Calculating the spectrum asymptote for locally isotropic turbulence in the viscous range," Zh. Éksp. Teor. Fiz., 93, No. 6 (1987).
8. G. B. Whitham, Linear and Nonlinear Waves, Wiley, New York (1974).